TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 261, Number 2, October 1980

THE RAYLEIGH-SCHRÖDINGER EXPANSION OF THE GIBBS STATE OF A CLASSICAL HEISENBERG FERROMAGNET

BY

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ABSTRACT. The equilibrium Gibbs state of a classical Heisenberg ferromagnet is a probability measure on an infinite product of spheres. The Kirkwood-Salsburg equations may be iterated to produce a convergent high temperature expansion of this measure about a product measure. Here we show that this expansion may also be obtained as the Rayleigh-Schrödinger expansion of the ground state eigenvector of a differential operator. The operator describes a Markovian time evolution of the ferromagnet.

1. Introduction. The simplest statistical mechanical systems are magnets. We think of a spin variable associated with each molecular site in a crystal. The properties of such a crystal should be simpler in the limit of infinite volume, since this is the proper description of bulk matter, independent of size and boundary effects. Thus we may as well take the crystal to be infinite in the first place.

In the classical Heisenberg model the sites consist of the set Z^ν of integer lattice points in ν-dimensional space. (Of course $\nu = 3$ is most important for statistical mechanics.) The unit sphere S in n-dimensional Euclidean space, $n \geqslant 1$, describes the possible spin directions at a single site. (Here again $n = 3$ is most natural.) The space of possible configurations of the model ferromagnet is the product space $\Sigma = (S)^{Z^\nu}$. A point σ in Σ thus assigns to each j in the crystal Z^ν a spin direction σ_j in S. Note that the Ising model is the case $n = 1$. From now on, however, we restrict our attention to the continuous spin case $n \geqslant 2$.

A state μ is a probability measure on Σ. If f is a measurable function on Σ, its expectation will be denoted $\langle \mu, f \rangle$. Thus f may be thought of as a random variable. One state of interest is the product $\mu_0 = (\omega)^{Z^\nu}$ of normalized surface measures ω on the spheres. In this state the vector-valued random variables σ_j are independent. This corresponds to the situation at infinite temperature.

We are primarily interested in the Gibbs state of equilibrium statistical mechanics for fixed finite temperature. We let $\beta \geqslant 0$ be the inverse temperature. For $\beta > 0$ we expect the spins σ_j to have a tendency to align. Below a certain critical temperature, that is for β sufficiently large, we might expect a variety of Gibbs states, corresponding to different directions of magnetization of the crystal as a whole. Thus we expect a qualitative change at a certain critical temperature. This leads us to expect that the Gibbs state will be analytic in temperature only in a

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Received by the editors September 19, 1979.
AMS (MOS) subject classifications (1970). Primary 82A05; Secondary 60K35, 47A55.
Key words and phrases. Heisenberg ferromagnet, diffusion in infinite dimensions, Gibbs state, Kirkwood-Salsburg equations, high temperature expansion.

0002-9947/80/0000-0461/$03.25

neighborhood of infinite temperature (that is $\beta = 0$) down to the critical temperature.

The formal expression for the energy of a configuration σ is given by a sum of scalar products of spins at nearest neighbor sites in the crystal by

$$H = -\frac{1}{2} \sum_j \sum_{|k-j|=1} \sigma_j \cdot \sigma_k. \tag{1.1}$$

Thus when adjacent spins are aligned the energy is lower than when they are pointing in different directions. The Gibbs state is given formally by $\langle \mu, f \rangle = \langle \mu_0, \rho f \rangle / Z$, where $\rho = \exp(-\beta H)$ and $Z = \langle \mu_0, \rho \rangle$ is a normalization constant. Of course these expressions involving divergent sums over the entire crystal are highly ambiguous.

The rigorous definition of Gibbs state is usually in terms of specified conditional probabilities of configurations of spins in a finite subset of the crystal as functions of the spins outside the subset. These equations may be iterated to give Kirkwood-Salsburg type expansions. Actually, in the present context the natural expansion is the one introduced by Gruber and Merlini [3] for the Ising model. This was generalized to classical Heisenberg models by Israel [5]. These techniques show that the Gibbs state is indeed uniquely determined for small β. We know by the work of Fröhlich, Simon, and Spencer [2] that it is not unique for $\nu \geqslant 3$ and large β. Magnets magnetize at low enough temperature.

In this paper we give a definition of an equilibrium state μ as a normalized eigenvector of an operator in the space of measures $C(\Sigma)^*$. (We do not discuss here the question of whether this definition is rigorously equivalent to the usual definition of Gibbs state.) We show that the expansion of μ about μ_0 in powers of β is simply the Rayleigh-Schrödinger expansion of the eigenvector. We also prove that it converges for small β. The analysis is complicated by the need to introduce a different Banach space for the proof of convergence. This seems to be a problem with all Kirkwood-Salsburg type expansions; convergence in the usual norm on measures is just too much to hope for. In the work of Gruber and Merlini [3] the space is defined by a supremum of the absolute values of the measure on group characters. Israel [5] extended the analysis to compact groups and their homogeneous spaces. Our use of spherical harmonics is similar to that in the paper of Israel. The difference is that he deals with multiplication operators and exploits the Banach algebra property of functions with absolutely convergent expansions in spherical harmonics. We use differential operators and have to deal with their unbounded nature. The advantage of the present approach is that it exhibits the connection of the expansions in statistical mechanics with the traditional sort of eigenvalue problem.

There is already considerable work in this direction for the Ising model. In this case the operators are infinite matrices which generate Markov processes. Holley and Stroock [4] have studied high temperature expansions for the equilibrium states and discussed the equivalence with Gibbs states. In the present work, however, the operators are differential operators in infinitely many dimensions which generate

Markov diffusion processes. These continuous spin diffusion processes have been shown to exist in previous work [**1**] and this paper is a first study of their properties.

2. The zero eigenvalue problem. The definition of equilibrium state in this section is as the solution of a certain eigenvalue problem. The analysis is simplified by the fact that the eigenvalue is known to be zero. Thus we need only solve for the eigenvector. As a consequence, the ordinarily rather complicated Rayleigh-Schrödinger expansion will take the form of a geometric series.

The space of spin configurations is $\Sigma = S^{Z^\nu}$. For each j in Z^ν there is a projection $\pi_j: \Sigma \to S$ defined by $\pi_j \sigma = \sigma_j$. If h is a function on S, then $\pi_j^* h$ is the function lifted to Σ defined by $\pi_j^* h(\sigma) = h(\pi_j \sigma) = h(\sigma_j)$. An analogous map π_j^* may be used to lift differential forms from S to Σ.

If f is a function on Σ, and j is in Z^ν, we may fix the values σ_k for $k \neq j$ and regard f as a function of σ_j in S. We may take the differential and then lift back to Σ. This defines the jth partial differential $d_j f$.

The configuration space Σ is a product of spheres. The Laplace operator acting on functions on the $n - 1$ sphere S, $n \geqslant 2$, is defined to be $\Delta = \nabla g^{-1} d$, where d is the differential and ∇ is the covariant differential and g^{-1} is the inverse metric tensor which converts this into a scalar. The jth partial Laplace operator acting on functions on Σ is defined to be $A_j = \Delta_j = \nabla_j g^{-1} d_j$, where the partial differentials are taken with respect to σ_j. The Laplace operator acting on functions on Σ is then $A = \Sigma_j A_j$. We may regard this as the infinite-dimensional Laplace operator $A = \Delta = \nabla g^{-1} d$.

This Laplace operator $A = \Delta$ is certainly well defined on smooth functions that depend on only finitely many spins. It is natural to extend A to a somewhat larger class of functions on Σ by requiring that A be a closed operator in the Banach space $C(\Sigma)$ of continuous functions on Σ. Since $\exp(tA) = \exp(t\Sigma_j A_j) = \Pi_j \exp(tA_j)$ is well defined as an operator on $C(\Sigma)$ for $t \geqslant 0$, A is in fact the infinitesimal generator of a Markovian semigroup of operators on $C(\Sigma)$. In fact, it corresponds to a Brownian motion acting independently on each spin. To get something more interesting, we need some kind of interaction between spins.

The Hamiltonian H is supposed to provide this interaction, but it is not a well-defined function on Σ. However the partial differentials

$$d_j H = -\sum_{|k-j|=1} d_j(\sigma_k \cdot \sigma_j) \tag{2.1}$$

are sums with 2ν terms and so are well-defined differential forms. In fact the differential $dH = \Sigma_j d_j H$ is well defined on tangent vectors to Σ that represent changes at only finitely many sites, since then only finitely many summands contribute. Thus local energy changes dH are available to define the interaction.

We may thus introduce operators $B_j = -\beta d_j H g^{-1} d_j$ which differentiate the jth spin and define $B = \Sigma_j B_j$ as the interaction operator. We may think of this as $B = -\beta dHg^{-1}d$. It is a first order differential operator, defined on smooth functions on Σ that depend on only finitely many spins. Notice that the term B_j depends not only on σ_j but on the σ_k of the 2ν neighboring spins.

If we write θ_{kj} for the angle from σ_k to σ_j, then

$$d_j H = -\sum_{|k-j|} d_j \cos \theta_{kj} = \sum_{|k-j|=1} \sin \theta_{kj} d_j \theta_{kj}. \tag{2.2}$$

Define $\partial_j/\partial\theta_{kj} = d_j\theta_{kj} g^{-1} d_j$, where the inverse metric tensor g^{-1} converts the form $d_j\theta_{kj}$ and the differential d_j into a scalar. Then

$$B_j = \beta \sum_{|k-j|=1} d_j(\sigma_k \cdot \sigma_j) g^{-1} d_j = -\beta \sum_{|k-j|=1} \sin \theta_{kj} \partial_j/\partial\theta_{kj}. \tag{2.3}$$

The flow defined by B_j tries to bring the jth spin into alignment with its neighbors.

The operator $A + B$ is the sum of Brownian motion and aligning components. It may also be regarded as a closed operator in the Banach space $C(\Sigma)$. It has been shown [**1**] that $A + B$ generates a Markovian semigroup $\exp(t(A + B))$, $t \geqslant 0$, of operators on $C(\Sigma)$. Since $(A + B)1 = 0$, the semigroup leaves 1 invariant. The adjoint semigroup $\exp(t(A + B))^*$ acting on the space $C(\Sigma)^*$ of measures describes a time evolution of probability measures on Σ. Such a probability measure μ is called an equilibrium state if $\exp(t(A + B))^*\mu = \mu$ for all $t \geqslant 0$. It is easy to see that this is equivalent to $(A + B)^*\mu = 0$. This is the equation for an eigenvector with zero eigenvalue. It has been shown [**1**] that equilibrium states exist. Thus the main question is uniqueness.

These measures μ will not usually have a density with respect to the product measure μ_0. But if we introduce a formal density ρ, the zero eigenvalue equation becomes

$$A^*\rho + B^*\rho = \nabla g^{-1}(d\rho + \beta dH\rho) = 0. \tag{2.4}$$

This has an obvious solution $d\rho/\rho = -\beta dH$ or $\rho = \exp(-\beta H)/Z$. Thus, on a formal level, Gibbs states appear to be equilibrium states.

The zero eigenvalue equation has an obvious rigorous solution for $\beta = 0$, namely $A^*\mu_0 = 0$. In the general case it is possible to transform the equation into a form suitable for iteration. For this we need to invert A. This can be done using its eigenfunctions.

Let p be an integer $\geqslant 0$. Let H_p be the space of spherical harmonics on the sphere S of degree p. This is the eigenspace of Δ with eigenvalue $\lambda(p) = -p(p + n - 2)$. Let ω be the normalized surface measure on the sphere S. Then Δ is selfadjoint in the Hilbert space $L^2(S, \omega)$ and the direct sum of the spaces H_p is $L^2(S, \omega)$.

The Hilbert space $L^2(S, \omega)$ is isomorphic to the subspace $\pi_j^* L^2(S, \omega)$ of the Hilbert space $L^2(\Sigma, \mu_0)$ obtained by lifting functions from S to Σ. Let l be a function from Z^ν to integers $\geqslant 0$ which is zero except at finitely many points. Define $H_l = \bigotimes_j \pi_j^* H_{l_j}$ as the space of linear combinations of products of elements of the spaces $\pi_j^* H_{l_j}$. Then H_l is an eigenspace of $A_j = \Delta_j$ with eigenvalue $\lambda(l_j)$. The direct sum of the H_l is the entire Hilbert space $L^2(\Sigma, \mu_0)$.

The eigenvalues of $A = \Sigma_j A_j$ are $\lambda(l) = \Sigma_j \lambda(l_j)$. These are nonzero except on the one-dimensional space H_0 spanned by 1. Thus if h is in the orthogonal complement, that is $\langle \mu_0, h \rangle = 0$, then we may define $A^{-1}h$ in the orthogonal complement in such a way that $AA^{-1}h = h$. We make the convention that $A^{-1}1 = 0$. Thus we may write an arbitrary f as $f = h + \langle \mu_0, f\rangle 1$ and obtain

$$AA^{-1}f = AA^{-1}h = h = f - \langle \mu_0, f\rangle 1. \tag{2.5}$$

PROPOSITION 1. *The equations* $(A + B)^*\mu = 0$ *and* $\langle \mu, 1\rangle = 1$ *imply that* $\langle \mu, f\rangle + \langle \mu, BA^{-1}f\rangle = \langle \mu_0, f\rangle$ *for all* f *in the space* D *of finite linear combinations of elements of the various spaces* H_l.

This proposition follows from the fact that the operator A^{-1} leaves D invariant and A and B are defined on all of D. Thus

$$\begin{aligned} 0 &= \langle (A + B)^*\mu, A^{-1}f\rangle = \langle \mu, (A + B)A^{-1}f\rangle \\ &= \langle \mu, f - \langle \mu_0, f\rangle 1 + BA^{-1}f\rangle = \langle \mu, (1 + BA^{-1})f\rangle - \langle \mu_0, f\rangle. \end{aligned}$$

Since D is dense in $C(\Sigma)$, any probability measure is determined by its restriction to D. Thus in order to prove uniqueness, it is sufficient to solve the above equation for functionals on D. If μ is a probability measure, we may define $A^{-1*}B^*\mu$ by $\langle A^{-1*}B^*\mu, f\rangle = \langle B^*\mu, A^{-1}f\rangle = \langle \mu, BA^{-1}f\rangle$ for f in D. Then $A^{-1*}B^*\mu$ is a functional on D. The equation in Proposition 1 then takes the form

$$\mu + A^{-1*}B^*\mu = \mu_0. \tag{2.6}$$

The Rayleigh-Schrödinger expansion in this case is just the geometric series expansion of $\mu = (1 + A^{-1*}B^*)^{-1}\mu_0$.

3. The Rayleigh-Schrödinger expansion. The simplest case of continuous spin is when $n = 2$ and the $n - 1$ spheres S are circles. This is called the plane rotor. The operators A and B are $A = \sum_j \partial^2/\partial\theta_j^2$ and $B = -\beta\sum_j \sum_{|k-j|=1} \sin(\theta_j - \theta_k)\partial/\partial\theta_j$. The product state μ_0 satisfies $A^*\mu_0 = 0$ and the equilibrium state μ satisfies $(A + B)^*\mu = 0$.

THEOREM 1. *For the plane rotor in* ν *dimensions with nearest neighbor interaction the Rayleigh-Schrödinger expansion for the equilibrium state* μ *in powers of* β *about* μ_0 *converges for* $2\nu\beta < 1$.

Since the expansion determines the value of μ on a dense set in $C(\Sigma)$ within its radius of convergence, this theorem proves that the equilibrium state is unique at high temperature. The upper bound on the critical temperature is the number 2ν of nearest neighbors.

We proceed to the proof. The space of spherical harmonics of degree l_j on the jth circle is spanned by $\exp(il_j\theta_j)$ and $\exp(-il_j\theta_j)$ for $l_j \geq 1$ and by 1 for $l_j = 0$. It is more convenient in this case to use instead the one-dimensional spaces spanned by $\exp(il_j\theta_j)$ for l_j an arbitrary integer. Thus for each function $l\colon Z^\nu \to Z$ which is zero except at finitely many points, we define $e_l = \exp(il\theta) = \exp(i\sum_j l_j\theta_j) = \prod_j \exp(il_j\theta_j)$. The Fourier coefficients of a measure μ on $\Sigma = S^{Z^\nu}$ are defined to be $\hat{\mu}(l) = \langle \mu, e_l\rangle$. We define a norm $\|\mu\| = \|\hat{\mu}\|_\infty$. We shall see that

$$\|A^{-1*}B^*\mu\| \leq 2\nu\beta\|\mu\| \tag{3.1}$$

in this norm.

We must estimate

$$|\langle A^{-1*}B^*\mu, e_l\rangle| = |\langle \mu, BA^{-1}e_l\rangle| \leq \sum_j |\langle \mu, B_jA^{-1}e_l\rangle|. \tag{3.2}$$

We know that $A^{-1}e_l = -1/(\Sigma_j l_j^2)e_l$ for $l \neq 0$. Furthermore, B_j is the sum of 4ν terms of the form $(\pm i\beta/2)\exp(\mp i\theta_k)\exp(\pm i\theta_j)\partial/\partial\theta_j$. It follows that $B_jA^{-1}e_j$ is the sum of 4ν terms of the form $(\pm\beta/2)l_j/(\Sigma_j l_j^2)e_{l'}$. Furthermore $|\langle \mu, e_{l'}\rangle| = |\hat{\mu}(l')| \leq \|\hat{\mu}\|_\infty$. Thus

$$|\langle \mu, BA^{-1}e_l\rangle| \leq 2\nu\beta\Big(\sum_j |l_j|\Big)\Big/\Big(\sum_j l_j^2\Big)\|\hat{\mu}\|_\infty \leq 2\nu\beta\|\mu\|. \tag{3.3}$$

This estimate shows that if $2\nu\beta < 1$, then $A^{-1*}B^*$ is an operator with norm < 1. Thus the zero eigenvalue equation has a convergent geometric series solution. This proves the theorem.

REMARK. The reason why the $\exp(il\theta)$ are such nice basis functions is that they are eigenvectors of the generator $T = \Sigma_j\, \partial/\partial\theta_j$ of simultaneous rotations. Since B_j commutes with T, it leaves each eigenspace of T invariant. Thus the eigenvalue $i\Sigma_j\, l_j$ of T is preserved by B_j. At high temperature it follows from the expansion that $\langle \mu, e_l\rangle \neq 0$ only for $\Sigma_j\, l_j = 0$. Thus there is no symmetry breaking in perturbation theory. Of course this can fail at low temperature when μ is not unique. A choice of μ may single out a particular direction, even though the Hamiltonian is rotation invariant.

THEOREM 2. *For the classical Heisenberg model in ν dimensions with nearest neighbor interaction the Rayleigh-Schrödinger expansion of the equilibrium state μ in powers of β about μ_0 converges for* $8\nu\beta < 1$.

Again this implies that the state is unique at high temperature. The proof follows the same lines as the proof for the plane rotor. However when $n \geq 3$ the rotation group that acts on the $n - 1$ sphere S is no longer commutative, and this complicates the analysis. We must deal with spherical harmonics.

We begin with some general facts about spherical harmonics. The book of Stein and Weiss [7] is an excellent reference. Let S be the unit $n - 1$ sphere in $\mathbf{R}^n$ with normalized surface measure ω. For each integer $p \geq 0$, let A_p be the space of harmonic polynomials that are homogeneous of degree p. The elements y of A_p are characterized by the two equations $\Delta_n y = 0$ and $r\partial y/\partial r = py$ of Laplace and Euler. Here Δ_n is the Laplacian in n-dimensional space. It is related to the angular Laplacian Δ by

$$\begin{aligned}\Delta_n &= (1/r^{n-1})\partial/\partial r(r^{n-1})\partial/\partial r + (1/r^2)\Delta \\ &= (1/r^2)\{(r\partial/\partial r)^2 + (n-2)r\partial/\partial r + \Delta\}.\end{aligned} \tag{3.4}$$

Let H_p, the space of spherical harmonics of degree p, consist of the restrictions of the elements of A_p to S. It follows that $\Delta Y = \lambda(p)Y$ where $\lambda(p) = -p(p + n - 2)$ for Y in H_p. The direct sum of the spaces H_p is $L^2(S, \omega)$.

Let e be a fixed vector in S. The basic object under consideration is the function on S defined by $s_e = s \cdot e = \cos\theta$, where θ is the angle from s in S to the fixed point e in S. Its differential is $ds_e = d\cos\theta = -\sin\theta d\theta$ and the corresponding differential operator is $ds_e g^{-1}d = -\sin\theta d\theta g^{-1}d = -\sin\theta\,\partial/\partial\theta$.

LEMMA 1. *The operator $ds_e g^{-1}d$ restricted to the space H_p of spherical harmonics of degree p has norm bounded by $(-\lambda(p))^{1/2}$.*

To prove this, let $x_e = x\cdot e$ be the linear functional on $\mathbf{R}^n$ whose restriction to S is s_e. Then the restriction of dx_e is ds_e. Since dx_e has length one, $dx_e g^{-1}dx_e = 1$, it follows that ds_e has length less than one, $ds_e g^{-1}ds_e \leq 1$. By the Schwarz inequality $|ds_e g^{-1}dY|^2 \leqslant d\bar{Y}g^{-1}dY$. Hence

$$\langle\omega, |ds_e g^{-1}dY|^2\rangle \leqslant \langle\omega, d\bar{Y}g^{-1}dY\rangle = -\langle\omega, \bar{Y}\nabla g^{-1}dY\rangle = -\langle\omega, \bar{Y}\Delta Y\rangle. \quad (3.5)$$

If Y is in H_p, the right-hand side is $-\lambda(p)\langle\omega, |Y|^2\rangle$. This estimate proves the lemma.

LEMMA 2. *The operator multiplication by s_e and the operator $ds_e g^{-1}d$ both send H_p into $H_{p-1}\oplus H_{p+1}$.*

For the proof we use the linear functional $x_e = x\cdot e$ on $\mathbf{R}^n$. Let Y in H_p be the restriction of y in A_p to S. Then $x_e y$ is a polynomial of degree $p+1$. Thus its restriction $s_e Y$ is a sum of spherical harmonics of degree $\leq p+1$. By the same reasoning, if Z is in H_q for some $q \leqslant p-2$, then $s_e Z$ is a sum of spherical harmonics of degree $\leqslant p-1$. But then $\langle\omega, s_e \overline{Y}Z\rangle = \langle\omega, \overline{Ys_e Z}\rangle = 0$, so $s_e Y$ is orthogonal to H_q. This implies that $s_e Y$ must be in the direct sum of H_{p-1}, H_p, and H_{p+1}. Under space inversion y goes to $(-1)^p y$ and $x_e y$ goes to $(-1)^{p+1}y$. Thus $x_e y$ has opposite parity from y and so its restriction $s_e Y$ has no component in H_p. The conclusion is that $s_e Y$ must be in $H_{p-1}\oplus H_{p+1}$.

Now write $x' = x/r$ and $x'_e = x'\cdot e = x_e/r$, where $r^2 = x\cdot x$. Then $x_e = rx'_e = r\cos\theta$, where θ is the angle between x and the unit vector e. The decomposition of dx_e into radial and tangential parts is

$$dx_e = x'_e dr + r dx'_e = \cos\theta dr - r\sin\theta d\theta. \quad (3.6)$$

The corresponding decomposition of the vector field obtained by applying the inverse g^{-1} of the metric tensor g on $\mathbf{R}^n$ gives

$$\partial/\partial x_e = x'_e\partial/\partial r + r dx'_e g^{-1}d = \cos\theta\,\partial/\partial r - (1/r)\sin\theta\,\partial/\partial\theta. \quad (3.7)$$

If y is in A_p, this gives

$$r dx'_e g^{-1}dy = -(1/r)\sin\theta\,\partial y/d\theta = \partial y/\partial x_e - (1/r^2)p x_e y. \quad (3.8)$$

Notice that $\partial y/\partial x_e$ is in A_{p-1}. Let Y and W in H_p and H_{p-1} be the restrictions of y and $\partial y/\partial x_e$. When we restrict to S we obtain

$$ds_e g^{-1}dY = -\sin\theta\,\partial Y/\partial\theta = W - p s_e Y. \quad (3.9)$$

Since $s_e Y = \cos\theta Y$ is already known to be in $H_{p-1}\oplus H_{p+1}$, this proves the lemma.

REMARK. Lemma 2 as stated above does not give the explicit form of the decomposition into components in H_{p-1} and H_{p+1}, but only states a selection rule to the effect that no other components can occur. This is in the spirit of the Wigner-Eckart theorem [**6**]. However the decomposition is given by

$$(2l + n - 2)x_e y = u + r^2 \partial y / \partial x_e. \tag{3.10}$$

It may be verified that if y is in A_p, then u is in A_{p+1}. Thus $(2l + n - 2)s_e Y = U + W$, where U in H_{p+1} and W in H_{p-1} are explicitly known.

Now we are ready to complete the proof for the classical Heisenberg model. We have $\Sigma = S^{Z'}$ and spaces $H_l = \bigotimes_j \pi_j^* H_{l_j}$ decomposing $L^2(\Sigma, \mu_0)$, where μ_0 is the product measure. The space H_{l_j} consists of the spherical harmonics of degree l_j on the sphere S.

A measure μ belonging to $C(\Sigma)^*$ has restrictions $\tilde{\mu}(l)$ to the spaces H_l. The restriction $\tilde{\mu}(l)$ is an element of the dual space H_l^* of the Hilbert space H_l and so has a Hilbert space norm $\|\tilde{\mu}(l)\|_2$. Let $\tilde{\mu}$ be the sequence of such restrictions and $\|\tilde{\mu}\|_2$ the corresponding numerical sequence of norms. We define the norm $\|\mu\| = \|\,\|\tilde{\mu}\|_2\|_\infty$ as the supremum of these norms.

We shall see that $\|A^{-1*}B^*\mu\| \leqslant 8\nu\beta\|\mu\|$ in this norm. Let Y be in H_l. We estimate

$$|\langle A^{-1*}B^*\mu, Y\rangle| = |\langle \mu, BA^{-1}Y\rangle| \leqslant \sum_j |\langle \mu, B_j A^{-1} Y\rangle|. \tag{3.11}$$

We know that $A^{-1}Y = \lambda(l)^{-1}Y$, where $\lambda(l) = \Sigma_j \lambda(l_j)$. So we need only estimate $B_j Y$.

This term $B_j Y$ is β times a sum of 2ν terms of the form $d_j(\sigma_k \cdot \sigma_j)g^{-1}d_j Y$. It follows from Lemma 1 that the $L^2(S, \omega)$ norm of such a term, considered as a function of σ_j alone, is bounded by $(-\lambda(l_j))^{1/2}$ times the same norm of Y. Since this estimate is independent of σ_k, we may integrate over the remaining spins and conclude that the $L^2(\Sigma, \mu_0)$ norm of the term is bounded by the same constant times the $L^2(\Sigma, \mu_0)$ norm of Y.

Next we need to see to which spaces these terms belong. Let $e_1, \ldots, e_n$ be an orthonormal basis for $\mathbf{R}^n$. Then $\sigma_k \cdot \sigma_j = \Sigma_i (\sigma_k \cdot e_i)(e_i \cdot \sigma_j)$ and thus

$$d_j(\sigma_k \cdot \sigma_j) = \sum_i (\sigma_k \cdot e_i)\, d(e_i \cdot \sigma_j). \tag{3.12}$$

Thus

$$d_j(\sigma_k \cdot \sigma_j)g^{-1}d_j Y = \sum_i (\sigma_k \cdot e_i)\, d(e_i \cdot \sigma_j)g^{-1}d_j Y. \tag{3.13}$$

By Lemma 2, as a function of σ_k and σ_j this belongs to the direct sum of the four spaces $H_{l_k \pm 1} \otimes H_{l_j \pm 1}$. It follows that as a function on Σ it is a sum of four terms belonging to spaces $H_{l'}$. Since those spaces are orthogonal, each of these terms individually has $L^2(\Sigma, \mu_0)$ norm bounded by $(-\lambda(l_j))^{1/2}$ times the $L^2(\Sigma, \mu_0)$ norm of Y.

It follows from the above analysis that $B_j Y$ is a sum of 8ν terms belonging to various spaces $H_{l'}$, each with norm bounded by $\beta(-\lambda(l_j))^{1/2}\|Y\|_2$. Thus

$$|\langle \mu, B_j A^{-1} Y\rangle| \leqslant 8\nu\beta\big(-\lambda(l_j)\big)^{1/2}(-\lambda(l))^{-1}\|Y\|_2. \tag{3.14}$$

Since $\lambda(l) = \Sigma_j \lambda(l_j)$, we conclude that

$$|\langle \mu, BA^{-1}Y\rangle| \leqslant 8\nu\beta\|Y\|_2. \tag{3.15}$$

This is the required estimate.

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